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Influence of temperature fluctuations on equilibrium ice sheet volume

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The *Oer03* model

The *Oer03* model is introduced in Oerlemans (2003) – some details are briefly summarized here. The model is "highly parameterized" and coupled to the surrounding climate by the altitude of the runoff line. Effectively the model consists of three steps: 1) describing the shape of the ice sheet, 2) analytically integrating the mass balance over the ice sheet and 3), numerically integrating the resulting expression for dR/dt where R is the radius of the ice sheet; the volume V is then uniquely determined from R.

Above the runoff line the accumulation is constant, below the balance gradient is constant; this is illustrated in Fig. (1) in the main article. The ice sheet is axially symmetric and rests on a sloping bed; furthermore ice is assumed to be a perfectly plastic material (Oerlemans, 2003).

The parameters we use are shown in Table 1. We have kept most parameters fixed as compared to Oerlemans (2003) but changed a total of 7 values to crudely approximate Greenland – note that we do not claim to be able to make accurate predictions of the GrIS even with this parametrisation. The temperature $\overline{T} = 5.8$°C has been chosen so that no temperature anomaly (i.e. setting $T = 0$ in Eq. 2) gives a equilibrium volume of about 7m SLE, corresponding roughly to the GrIS (Church et al., 2013).

Steps 1 through 11 below describe the *Oer03* model setting used – these steps describe calculations performed at every time step that give an expression for

$$\frac{dR}{dt} = f(T,R); \tag{1}$$

dR/dt is then integrated using the Euler scheme with a time step of 1 year. We find that using a smaller time step size than this only produce negligible differences – see Figure 1 for an example. Since we describe an algorithm and not a derivation below, we will use left arrows for assignment.

1. We couple the ice sheet to the ambient temperature by introducing the following relationship between temperature and height of the equilibrium line (Oerlemans, 2008):

   $$h_{Eq} \leftarrow h_{E,0} + (T - \bar{T}) \cdot 1000/6.5. \tag{2}$$

   As in the main article, Eq. 2 represents an increase of the equilibrium line altitude of approximately 154 m °C^{-1}.

2. Equation 3 reflects that the accumulation rate will likely decrease for a large ice sheet (Eq. 20 in Oerlemans (2003)):

   $$A \leftarrow A_0 \mathrm{e}^{-R/R_C}. \tag{3}$$

3. Height of the runoff line (Eq. 15 in Oerlemans (2003)):

   $$h_R \leftarrow h_{Eq} + A/\beta. \tag{4}$$

4. Height of the bedrock where the ice sheet ends:

$$h_E \leftarrow d_0 - sR. \tag{5}$$

5. Location where the runoff line and the ice sheet surface meet (Eq. 17 in Oerlemans (2003)):

$$r_R \leftarrow R - (h_R - h_E)^2/\mu. \tag{6}$$

6. Check if the ice sheet extends into the sea, i.e. if $R > r_c$. If so, use Eq. (7) in Oerlemans (2003) to define the radial coordinate of the grounding line r_{gr}:

   – **if** $R > rc$:

$$r_{gr} = R - h_E^2/\mu. \tag{7}$$

7. If the radial coordinate of the runoff line is larger than of the grounding line, set runoff coordinate to grounding coordinate:

   – **if** $r_R > r_{gr}$:

$$r_R \leftarrow r_{gr}. \tag{8}$$

8. If the height of the runoff line is smaller than the height of ice sheet termination, set radial coordinate of the runoff line to radius of the ice sheet:

   – **if** $h_R < h_E$

$$r_R \leftarrow R. \tag{9}$$

9. If $R < r_c$ the ice sheet is continental. Equations 10 and 11 are included for numerical reasons.

   – **if** $R \le r_c$

     – **if** $r_R < 0$

$$r_R \leftarrow 0 \tag{10}$$

     – **if** $R < 1$

$$R \leftarrow 1 \tag{11}$$

     – Calculate total $dV/dt = B_{tot}$:

$$\begin{aligned} B_{tot} \quad & \leftarrow \pi A R^2 && (12)\\ & -\pi\beta\,(h_R - h_E)\left(R^2 - r_R^2\right) && (13)\\ & +\frac{4\pi\beta\mu^{1/2}}{5}\,(R - r_R)^{5/2} && (14)\\ & -\frac{4\pi\beta\mu^{1/2}}{3}\,R\,(R - r_R)^{3/2}\,. && (15)\\ & && (16) \end{aligned}$$

10. If $R > r_c$ the the ice sheet extends into the sea:

– **if** $R > r_c$

$$B_{tot} \leftarrow \pi A r_{gr}^2 \tag{17}$$

$$-\pi\beta\left(h_R - h_E\right)\left(r_{gr}^2 - r_R^2\right) \tag{18}$$

$$+\frac{4\pi\beta\mu^{1/2}}{5}\left(\left(R - r_R\right)^{5/2} - \left(R - r_{gr}\right)^{5/2}\right) \tag{19}$$

$$-\frac{4\pi\beta\mu^{1/2}}{3}\left(R\left(R - r_R\right)^{3/2} - R\left(R - r_{gr}\right)^{3/2}\right) \tag{20}$$

$$-2\pi r_{gr}\left(\frac{\rho_w}{\rho_i}\right) f\left(s r_{gr} - d_0\right)^2. \tag{21}$$

Here the last term corresponds to Eq. 19 in Oerlemans (2003) and is related to the flux across the grounding line.

11. Relationship between $\frac{dR}{dt}$ and B_{tot}, corresponding to Eq. 13 in Oerlemans (2003):

– **if** $R \leq r_c$

$$Q \leftarrow \pi\left(1 + \frac{\rho_i}{\rho_m - \rho_i}\right)\left(\frac{4}{3}\mu^{1/2}R^{3/2} - sR^2\right), \tag{22}$$

$$\frac{dR}{dt} \leftarrow B_{tot}/Q. \tag{23}$$

– **if** $R > r_c$

$$Q \leftarrow \pi\left(1 + \frac{\rho_i}{\rho_m - \rho_i}\right)\left(\frac{4}{3}\mu^{1/2}R^{3/2} - sR^2\right) \tag{24}$$

$$-2\frac{\rho_w}{\rho_m - \rho_i}\left(\pi s R^2 - d_0 R\right), \tag{25}$$

$$\frac{dR}{dt} \leftarrow B_{tot}/Q. \tag{26}$$

Integrating steps 1-11 yield a time series of the ice sheet radius. To convert to volume we use the following relations (Eqs. 9, 11 and 12 in Oerlemans (2003)); the volume of the continental part of the ice sheet:

$$V_{cont} = \frac{8\pi\mu^{1/2}}{15}R^{5/2} - \frac{1}{3}\pi s R^3. \tag{27}$$

In the case of the ice extending to the sea, the volume of the sea water replaced by ice:

$$V_{sea} = \pi\left(\frac{2}{3}s\left(R^3 - r_c^3\right) - d_0\left(R^2 - r_c^2\right).\right) \tag{28}$$

V_{sea} is set to zero if the ice does not extend to the sea and thus $R < r_c$. The total volume is given by:

$$V_{tot} = V_{cont}\left(1 + \frac{\rho_i}{\rho_m - \rho_i}\right) - \frac{\rho_w}{\rho_m - \rho_i}V_{sea}. \tag{29}$$

Name	Unit	Value	Notes
A_0	m ice yr^{-1}	1.0 †	Characteristic specific balance.
β	m ice yr^{-1}m^{-1}	0.005 †	Specific balance gradient.
c	m$^{1/2}$	2×10^6 †	Bed slope effect parameter.
C_R	m	5×10^5 †	e-folding radius for "desert effect" from large ice sheets; see Eq. 3
d_0	m	$= h_{E,0}$ *)	Undisturbed bed height at center of ice sheet.
h_{Eq}	m	See Eq. 2 *)	Height of equilibrium line.
$h_{E,0}$	m	1545 *)	Equilibrium line height at $T = 0$. Approximate 1990 - 2010 average (NOAA (2015), Fig 3.2a)
f	yr^{-1}	0.5 $^{+}$	Bulk flow parameter related to ice discharge.
μ_0	m$^{1/2}$	8 †	Bed slope effect parameter.
$\mu = \mu_0 + cs^2$			Equation 4 in Oerlemans (2003)
ρ_i	kg m^{-3}	900 $^{+}$	Density of ice.
ρ_w	kg m^{-3}	1025 $^{+}$	Density of sea water.
ρ_m	kg m^{-3}	3500 $^{+}$	Density of bedrock.
r_c	m	8×10^5 *)	Continental radius. Approximate width of Greenland.
r_{gr}	m	8×10^5 *)	Initial value – dynamical value in the model.
s	m/m	$d_0/r_c \approx 0.002$ *)	Bed slope.
$\bar{T}$	°C	5.8*)	Temperature offset.

Table 1. †: Suggested in Oerlemans (2003). $^{+}$: suggested in private communication with Hans Oerlemans. *): chosen by the present authors.

Time step size Δt used for numerical integration

To determine an adequate time step size Δt to use for numerically integrating Eq. 1, we first generate a time series of fluctuating temperatures $\{T_t\}$ as described by Eq. (8) in the main article. With $\{T_t\}$ as input, and a similar initial condition as the simulations shown in Fig. 2 in the article, Eq. 1 is numerically integrated for varying Δt using the Euler scheme. Δt is varied in such a way that the temperature is the same for each whole year, regardless of the time step size. The results of varying Δt from 0.01 year to 1 year are shown in Fig. 1. As the resulting graphs of the ice sheet volume $V(t)$ practically coincide, we consider a time step size of one year to be sufficient.

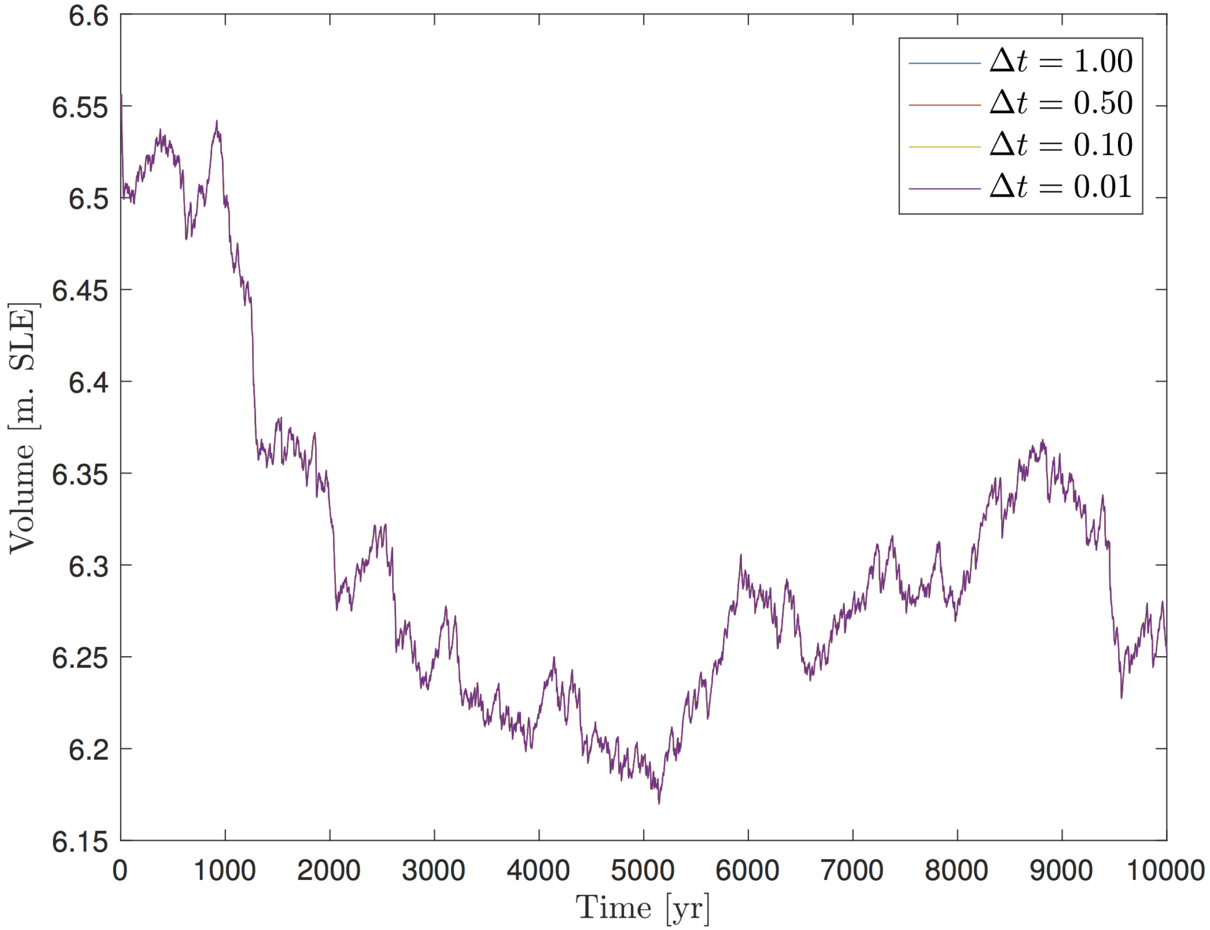

Figure 1. Varying the integration stepsize Δt from 1 year to 0.01 years for a simulation with $\overline{T} = 0$, such that the (random) fluctuating temperature T_t is the same for each whole year. A visual inspection confirms qualitatively that the graphs for varying Δt coincide and we do not further analyze the consequences of varying Δt.

Observed fluctuations in Greenland temperature

Surface temperature anomalies were obtained from (KNMI). We use the "Twentieth Century Reanalysis V2c" from the years 1851 to 2011 in a box spanning 68°N to 80°N and 25°W to 60°W. The raw data consists of monthly means and are shown in Figure 2 as the blue curve.

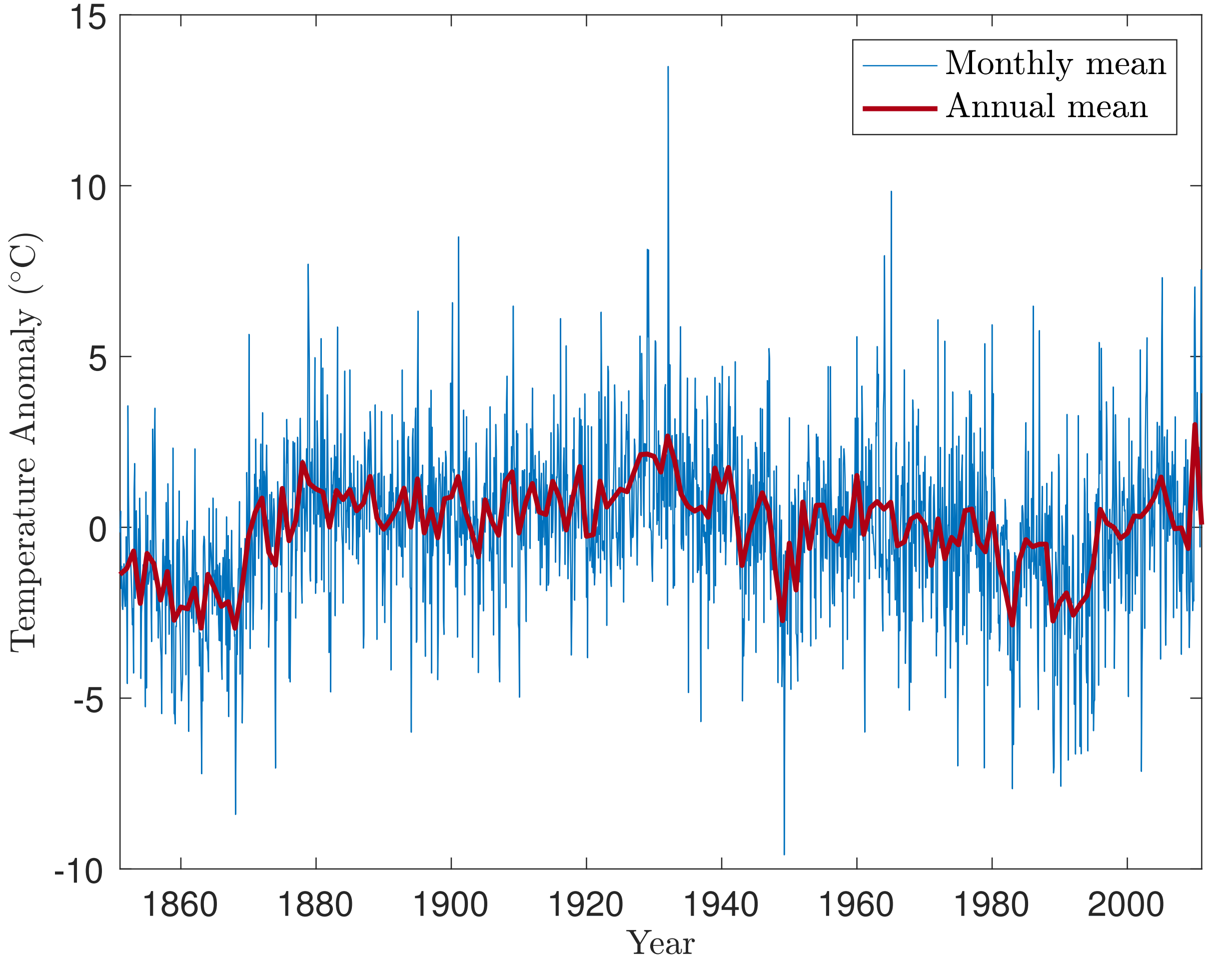

Figure 2. Reanalysis data showing monthly mean surface temperature anomaly (blue curve) over the area 68°N – 80°N, 25°W – 60°W covering a large part of Greenland. The red curve is the annual mean surface temperature anomaly; this has observed variance $\sigma_{T,\text{obs}} = 1.55$ °C^2.

We treat the temperature data as follows:

1. We calculate the annual mean (the red curve in Figure 2),

2. To the annual means we fit an autoregressive model of order 1 or an AR(1)-model,

3. The parameters from this model are used to generate artificial temperature time series $\{T_t\}$ that fluctuate in a way similar to the observed temperatures over Greenland.

An AR(1) model describing describing $\{T_t\}$ has the form

$$T_{t+1} = c + aT_t + \sigma_{AR} W_t. \tag{30}$$

where (c, a, σ_{AR}) are parameters to be determined and W_t is white noise with unit variance and zero mean. The parameters (a, σ_{AR}) are found using MATLAB's `estimate()`. We find

$$(a, \sigma_{AR}^2) \quad = (0.67, 0.85). \tag{31}$$

Evaluating neglected terms in Eq. (6)

In the text between Eqs. (4) and (5) in the main article we argue that the terms $\langle (V_t - \overline{V})^2 \rangle$ and $\langle (T_t - \overline{T})(V_t - \overline{V}) \rangle$ can be neglected for the Oer03 model since they tend to zero when the ice sheet approaches equilibrium. This can be seen in Fig. (3) where we evaluate these terms numerically for an ensemble of simulations with parameters identical to the simulations shown in Fig. 2 (main article, left). To construct one of the time series in Fig. (3), we first integrate the Oer03 model using a temperature time series $\{T_t\}$ with mean $\overline{T}$, and then find the steady state mean volume $\overline{V}$. Then, at each time step, the quantities in Fig. (3) are calculated. In each case, it is clear that the *mean* quantities $\langle (V(t) - \overline{V})^2 \rangle$ and $\langle (T(t) - \overline{T})(V(t) - \overline{V}) \rangle$ tend to zero.

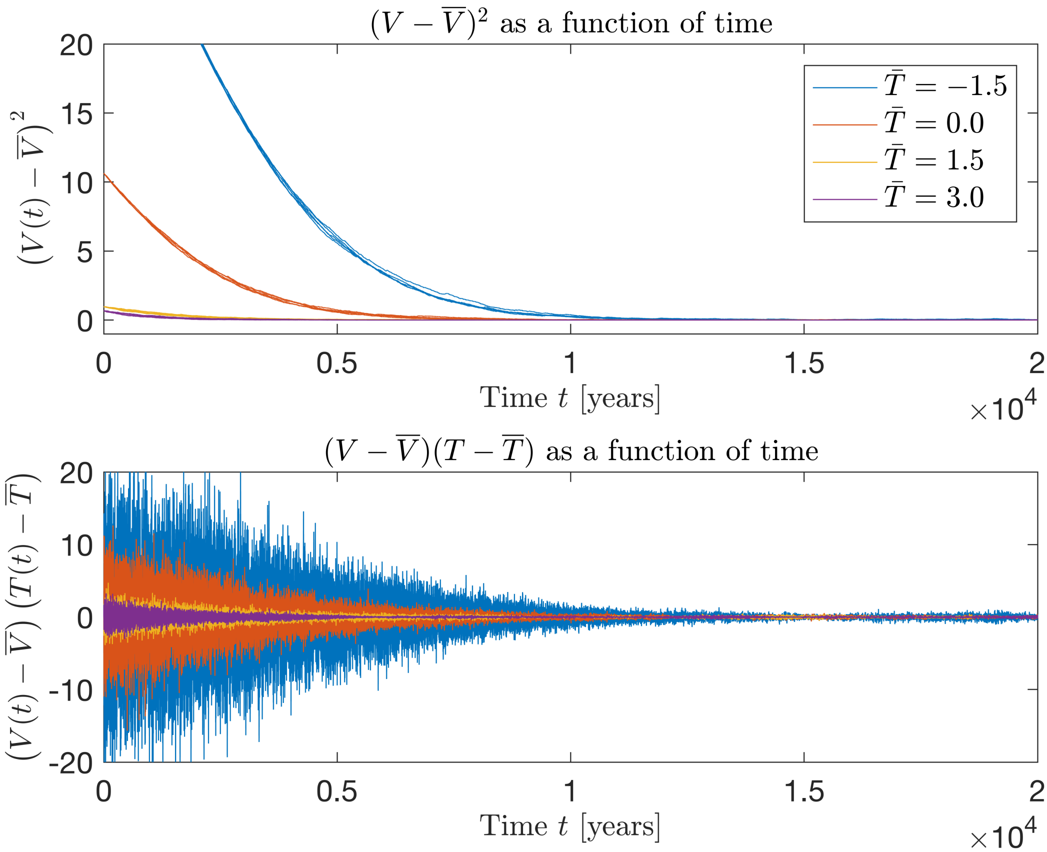

Figure 3. Evaluation of the terms dropped from Eq. 5 in the main article, for simulations with same parameters as in Fig. 2 (main article). It is clear that $\langle (V_t - \overline{V})^2 \rangle$ and $\langle (T_t - \overline{T})(V_t - \overline{V}) \rangle$ tend to zero.

Analysis of Robinson et al. (2012)'s data

We aim to estimate the effect of fluctuating temperatures on the results obtained by Robinson et al. (2012). Our aim is to find ΔT and ΔSMB as shown in Fig. 3 (main article, right) by fitting polynomials $\tilde{f}_{ij}(T)$ to the SMB as a function of warming temperature T (Eq. (10), main article).

Methodology

- In Robinson et al. (2012) the warming is ramped for the first 100 years, for numerical reasons. We wait until $t = 200$ years to extract SMB(T),

- Robinson et al. (2012) employ 9×11 values of two separate parameters deemed "equally likely" in their simulations, as well as 11 values for the warming, totalling 1089 individual ice sheet simulations,

- For each of the 99 parameter combinations, we fit a 3rd degree polynomial to the SMB(T), following Fettweis et al. (2013). We denote these fits $\tilde{f}_{ij}(T)$,

- We proceed as outlined in Eqs. (11) and (12) (main article),

- Finally we calculate 95% credible intervals for ΔT and ΔSMB for each value of T. This is done by fitting a density to the obtained ΔT and ΔSMB (using `MATLAB`'s `ksdensity()`) and calculating the interval containing 95% of the observations.

As stated in the conclusion in the main article, we must assume the ice sheet volume to be constant when calculating the $\tilde{f}_{ij}$'s since these functions represent Taylor expansions around a steady state (see Eqs. (3)-(6) in the main article). It is therefore relevant to measure to what extent the ice sheet volume varies. To measure this, we calculate, for each parameter pair in the simulations carried out by Robinson et al. (2012), the mean, maximum and minimum ice sheet volume for the range of 11 warming temperatures considered, denoted V_{mean}, V_{max} and V_{min}, respectively. We then calculate the quantity

$$V_{\text{diff, max}} = \frac{V_{\text{max}} - V_{\text{min}}}{V_{\text{mean}}} \times 100\% \quad (32)$$

for each parameter pair, giving us a measure of the maximum difference in ice sheet volume between the simulations, compared to the mean ice sheet volume, for that parameter pair. Histograms of $V_{\text{diff, max}}$ are shown in Fig. (4) for *all* the warmings considered in Robinson et al. (2012), and in Fig. (5) for warmings up to $4°$C.

The variations in ice sheet volume shown in Fig. (4) are substantial, up to 9.5%. To draw our conclusion, however, we only need to consider warming temperatures up to 4%. As shown in Fig. (5), the variation in this case is less than 3%.

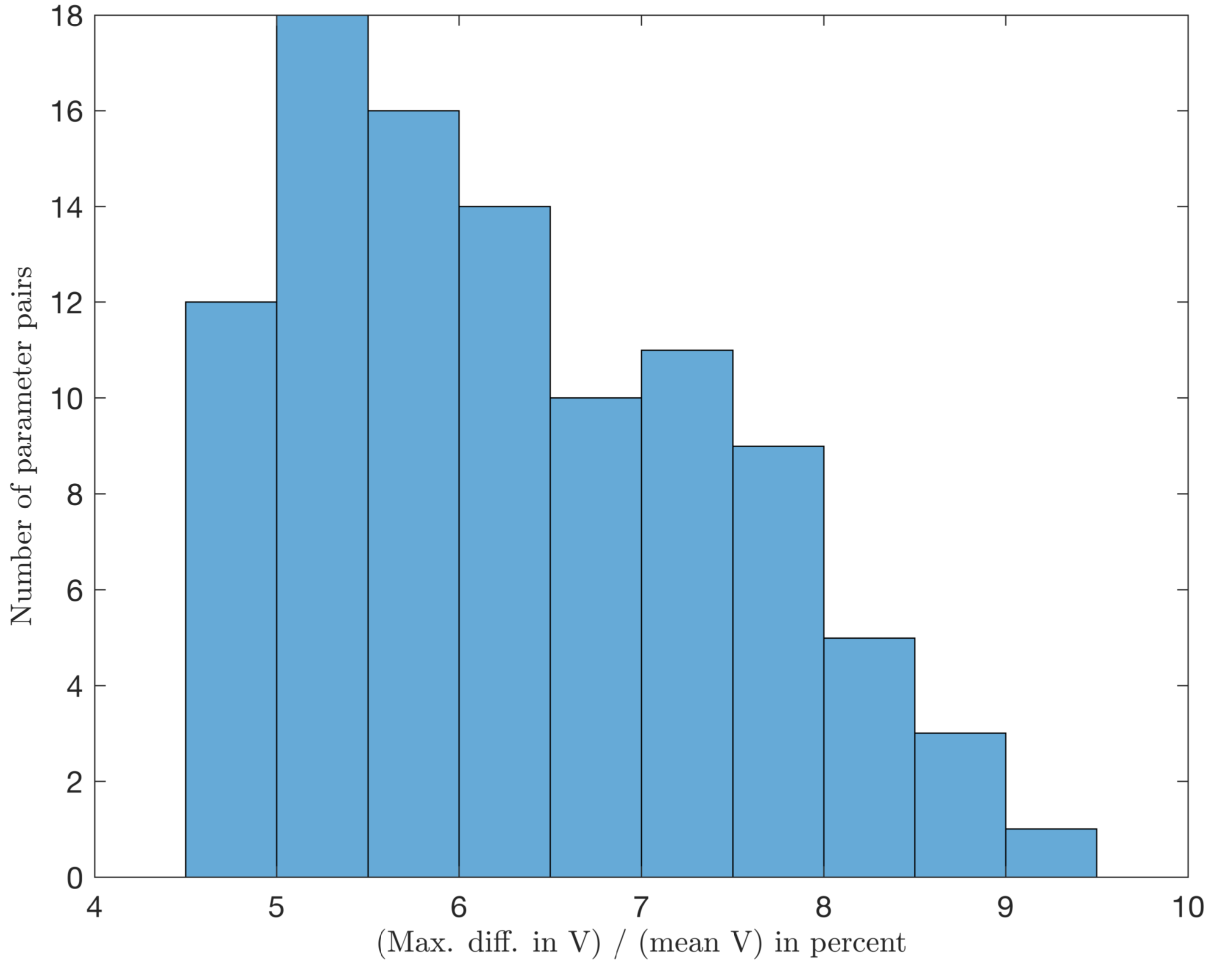

Figure 4. Histogram of the maximum difference in volume for different temperature anomalies divided by the mean volume $t = 200$ years in the data from Robinson et al. (2012), calculated for each parameter combination; in total there are 9×11 combinations of two separate parameters .

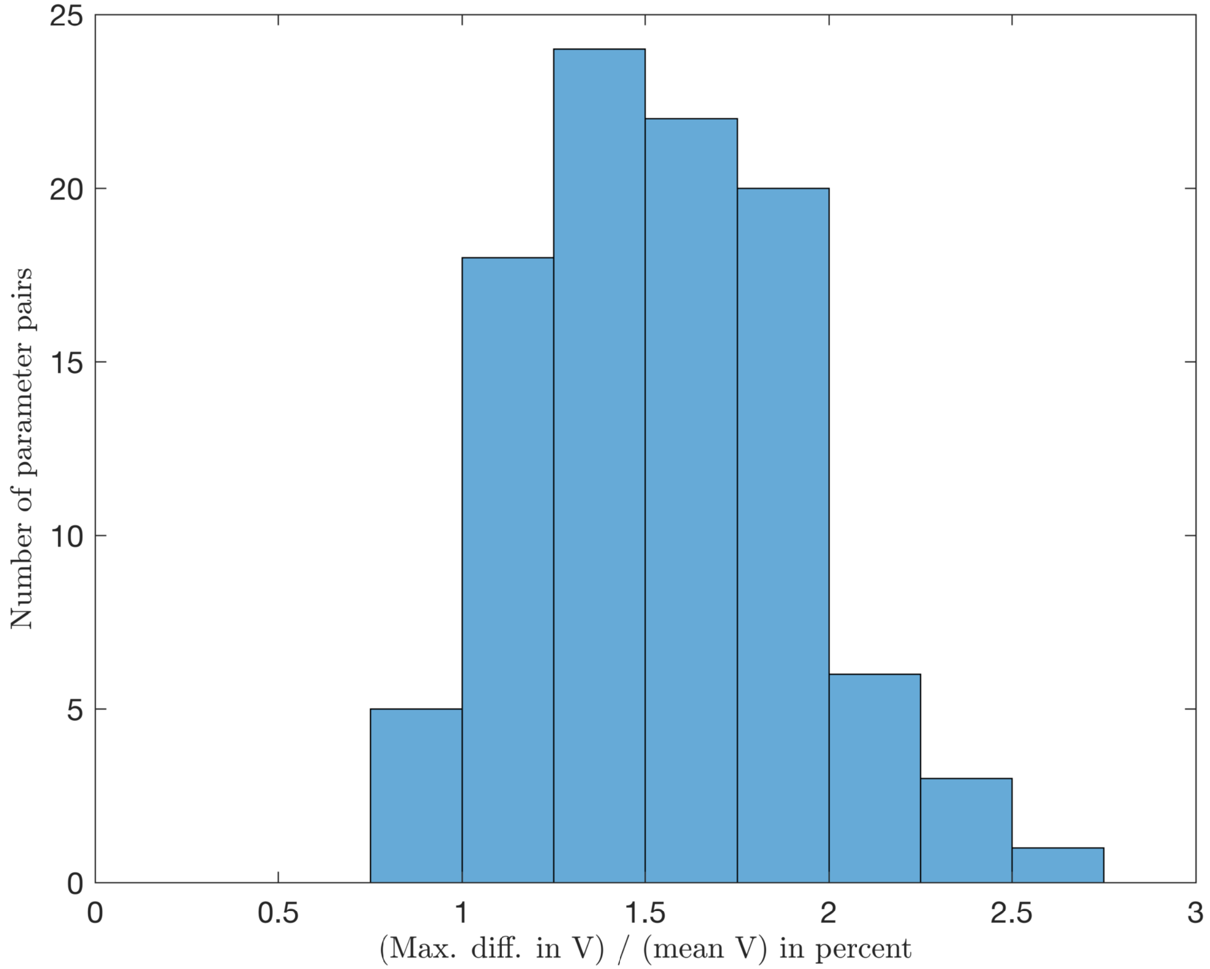

Figure 5. Same as Figure 4 but for a maximum warming of 4°C.

Conversion from $\mathbf{m^3}$ and Gt ice to meters sea level equivalent (m SLE)

We take the the surface area of the world's oceans A_{ocean} to be 361,900,000 km^2 (Eakins and Sharman, 2010), or 3.619×10^{14} m^2 and, as in Table 1, the density of ice ρ_i to be 900 kg m^{-3}, the density of sea water ρ_w to be 1025 kg m^{-3} and approximate the density of freshwater ρ_f as 1000 kg m^{-3} so that 1 Gt ice is equivalent to 10^9 m^3 freshwater.

To convert 1 m^3 ice to meters sea level equivalent:

$$1\text{m}^3 \text{ ice} = 1\text{m}^3 \frac{\rho_i/\rho_w}{A_{\text{ocean}}} \text{ m SLE} = 2.4262 \times 10^{-15} \text{ m SLE}. \tag{33}$$

To convert 1 Gt ice to sea level equivalent

$$10^9\text{m}^3 \text{ freshwater} = 10^9\text{m}^3 \frac{\rho_f/\rho_w}{A_{\text{ocean}}} \text{ m SLE} = 2.6958 \times 10^{-6}\text{m SLE}. \tag{34}$$

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