Physica D 466 (2024) 134215

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

On the representation of multiplicative noise in modeling Dansgaard–Oeschger events

Kolja Kypke *, Peter Ditlevsen

Physics of Ice, Climate and Earth, Niels Bohr Institute, University of Copenhagen, Tagensvej 16, 2200, Copenhagen, Denmark

ARTICLE INFO

Dataset link: https://www.iceandclimate.nbi.ku.dk/data/

Keywords:
Stochastic dynamical system
Dansgaard–Oeschger events
Multiplicative noise

ABSTRACT

The interpretation of multiplicative noise in a stochastic differential equation in the context of data-driven inverse modeling is discussed. Application to the well-known paleoclimate phenomenon of Dansgaard–Oeschger events leads to qualitatively different 'climate potentials' in the case of the Itô or the Stratonovich interpretation of the stochastic integral. While a physical model is endowed with an interpretation from construction, whether implicitly or explicitly, inverse models derived from data do not afford such a luxury. In this case, a physical model must accompany the mathematical model equation in order to be able to choose a stochastic interpretation. This case study illustrates the differences between the two representations of stochastic noise and demonstrates the need for input from physical constraints when constructing conceptual stochastic models of the observed climate records.

1. Introduction

One of the most famous examples of abrupt climate changes observed in the paleoclimatic record are the Dansgaard–Oeschger (D–O) events. The climate record of the Last Glacial Period (LGP), which spanned approximately 120 to 11 kiloyears before year 2000 (kyr b2k), is measured in the ice-cores of the Greenland ice sheet and marked by distinct and abrupt transitions between colder stadial and warmer interstadial periods [1]. These climatic changes are known as D–O events, and occurred about 24 times in the LGP. The D–O events correspond to approximately 10–15 Kelvin of warming in Greenland over the course of a few decades, with subsequently incremental cooling to the fully glacial conditions of the stadial [2]. While there is only direct evidence of D–O events in the LGP because the ice-core record of Greenland only extends to the end of the last interglacial period, they may not be unique to this time period. Coupling with Antarctic ice-cores [3], evidence in marine sediment cores [4] and speleothems [5] (see also references therein) suggest they may have occurred in previous glacial periods as well.

D–O events are interesting in the context of the present climate primarily because of their temporal scale. They are an example that the climate can change on time scales that could be of consequence in the near future, namely of decades to centuries. They are additionally intriguing because there is no universal agreement on their cause and transition mechanism. The transitions between stadial and interstadial states themselves may be externally forced [6–11], stochastic [12–17] or even both [18]. Possible important physical drivers for the transitions include change in sea ice [19,20], atmospheric carbon dioxide levels [21,22] or volcanic events [23]. Comprehensive models generally exhibit D–O events as oscillations [24,25] and transitions are not spontaneous. See also the review articles [26–28].

D–O events are difficult to simulate in complex models, thus a full understanding of their causes is still lacking. For this reason it is desirable to investigate them using low order dynamical systems models where the dynamics are in full view. Without a full understanding, a modeling strategy is to construct simplified models, optimizing parameters to best fit to observational data. This is an inverse modeling approach. Such conceptual models may be either data-derived [29–31] or constructed from physical principles [32,33]. Generally the data-derived inverse models of D–O events do not propose a physical mechanism whereas conceptual models developed from physical principles tautologically do.

1.1. Stochastic differential equation models of D–O events

This study derives a conceptual model from the data in an inverse modeling scheme. The model paradigm this study follows is to describe the climate proxy as the variable x of a stochastic differential equation

* Corresponding author.
E-mail address: kolja.kypke@nbi.ku.dk (K. Kypke).

https://doi.org/10.1016/j.physd.2024.134215
Received 25 September 2023; Received in revised form 15 February 2024; Accepted 16 May 2024
Available online 20 May 2024

(SDE):

$$dx = F(x)dt + \sigma(x)dW_t, \quad (1)$$

where the Wiener process $W_t = \int_0^t \xi_s ds$ has the properties $\langle \xi_t \rangle = 0$, $\langle \xi_{t_1}\xi_{t_2} \rangle = \delta(t_1 - t_2)$ and ξ_t is Gaussian distributed, i.e. ξ_t is a white noise Gaussian process. The crux of this methodology is to model the climate as being driven by some long timescale climate dynamics described by a function $F(x)$ along with a stochastic component that represents processes that occur on much faster time scales.

Observing the record showing two distinct climate states, the stadial and the interstadial (see Fig. 1), it is natural to consider the abrupt changes as being a transition from one stable state to the other stable state. The measured quantity in the ice core is δ^{18}O, a ratio of heavier ^{18}O isotopes to lighter ^{16}O, and is a proxy for local temperature. Correlation of ocean sediment records with the ice-core record [34] suggest that the Atlantic ocean is the source of the transitions. In this sense, the paleoclimate temperature variations are themselves a proxy of north Atlantic ocean circulation strength. A mechanism of bistablity in the north Atlantic is that of the thermohaline circulation with multiple modes of circulation [35]. Two regimes of flow, one with strong equator-to-pole meridional overturning circulation and one with weak circulation, correspond to warmer and colder Greenland temperatures respectively.

This bistability is seen in both conceptual models [36,37] as well as general circulation models [38], with more recent earth system models being able to reproduce spontaneous 'D–O-like' events [39]. A proposed mechanism for D–O events is such a bistable climate system, i.e. the ocean circulation, with a stochastic term, i.e. freshwater fluxes from atmospheric forcing via wind stress, surface heating, and precipitation, that causes transitions between the two states. Examples of studies that follow a similar framework involve methods such as models with non-Gaussian noise [12], Kalman filters [29], Gaussian mixture models or relaxation-oscillation models [15], Bayesian parameter inference [17, 30], and non-stationary potentials [16].

1.2. Additive noise

The majority of previous studies assume the intensity of the noise is constant or *additive* [12,16,17,29]. The Fokker–Planck equation associated with the additive noise SDE

$$dx = F(x)dt + \sigma dW_t \quad (2)$$

is

$$\partial_t P(x,t) = -\partial_x \left[F(x)P(x,t)\right] + \frac{\sigma^2}{2}\partial_x^2\left[P(x,t)\right]. \quad (3)$$

When the stationary probability distribution is obtained from the observed time series record, the potential that drives the dynamics of the additive noise SDE is obtained from the Fokker–Planck equation by having $\partial_t P = 0$:

$$U(x) = -\frac{\sigma^2}{2}\log[P(x)] \quad (4)$$

where $\frac{\partial U}{\partial x} = -F(x)$. The potential can be uniquely determined from the stationary probability density $P(x)$ up to a constant factor of the noise strength. Due to the monotonicity of the logarithm the maxima of the probability density function $P(x)$ coincide with the minima of the potential $U(x)$ and thus with the stable equilibria of the deterministic dynamics. Therefore the number of equilibria will always be identical to the number of maxima in probability density.

1.3. Multiplicative noise

From the paleoclimatic record it is observed that the intensity of the fast fluctuations constituting the noise as indicated in Eq. (1) depends on the climate state [40]. State dependent noise is termed *multiplicative* noise. In this case deriving the resulting potential is not completely straightforward. This is due to the fact that when integrating the noise term by way of generalized functions, the resulting Riemann–Stieltjes integral

$$\int_a^b G(t)dW(t) = \lim_{n\to\infty}\sum_{i=1}^{n} G(\tau_i)[W(t_i) - W(t_{i-1})], \quad (5)$$

where $\tau_i \in [t_{i-1}, t_i]$, has a different expected value depending on where in the interval τ_i is chosen. The two most common choices are the left endpoint, named the *Itô interpretation* [41], and the midpoint, named the *Stratonovich interpretation* [42]. Often the *α-convention* is used to designate the different interpretations, where the value of α in the interval $\tau_i = (1-\alpha)t_{i-1} + \alpha t_i$ is 0 for Itô and $1/2$ for Stratonovich. In theory, any value of $\alpha \in [0,1]$ is a valid choice for the stochastic integral, but these are by far the two most common. For a more general function $G\big(x(t)\big)$, the definition of the integral is

$$\int_a^b G\big(x(t)\big)dW(t) = \lim_{n\to\infty}\sum_{i=1}^{n} G\big(x^*(\tau_i)\big)[W(t_i) - W(t_{i-1})], \quad (6)$$

and the α-convention is $x^*(\tau_i) = (1-\alpha)x(t_{i-1}) + \alpha x(t_i)$.

Due to the difference of the stochastic integrals, Eq. (1) is incomplete and an interpretation of the noise term must be specified [43]. As a consequence the same SDE can result in different stochastic processes depending on whether the Itô or Stratonovich interpretation is applied. A corollary is that two different SDEs, one interpreted as Itô and the other as Stratonovich, can result in the same stochastic process. Thus solving the inverse problem of deriving the SDE, and especially the potential, from a stochastic realization requires a specification of the noise. Here we perform the derivation of the SDE from the data for both the Itô and Stratonovich integrals.

As SDEs they are distinguished by the notation $dx = F_I(x)dt + \sigma(x)\cdot dW_t$ for Itô and $dx = F_S(x)dt + \sigma(x)\circ dW_t$ for Stratonovich, where F_I and F_S are different potential functions. The associated Fokker–Planck equation using the α convention is

$$\partial_t P(x,t) = -\partial_x\left[\Big(F(x) + (1-\alpha)\sigma(x)\sigma'(x)\Big)P(x,t)\right] + \frac{1}{2}\partial_x^2\left[\sigma(x)^2 P(x,t)\right]. \quad (7)$$

As can be seen from this equation, a simple relation exists between the drift terms of the two interpretations,

$$F_S(x)dt + \sigma(x)\circ dW_t = \left[F_I(x) + \frac{1}{2}\sigma(x)\sigma'(x)\right]dt + \sigma(x)\cdot dW_t, \quad (8)$$

so any Stratonovich integral may be converted to an Itô integral and vice versa. This relation is especially useful when numerically integrating an SDE since the commonly used Euler–Maruyama method is only applicable to Itô SDEs. A Stratonovich SDE can be converted to one of the Itô type and integrated using the Euler–Maruyama method, or alternatively integrated with a predictor–corrector scheme such as a Heun method. Itô and Stratonovich integrals have some other differences as well, the most notable being that differentiation under the Itô interpretation requires the Itô lemma [41],

$$f\big(x(t)\big) = f\big(x(0)\big) + \int_0^t f'\big(x(s)\big)dx(s) + \frac{1}{2}\int_0^t f''\big(x(s)\big)ds. \quad (9)$$

On the other hand the Stratonovich interpretation uses the chain rule of regular calculus.

One previous study includes state-dependent noise in the form of a piecewise constant noise term, where the amplitude is a lower constant value in the interstadials than in the stadials [44]. However their state dependent noise function $\sigma(x)$ still has a derivative that is zero except for a single point, so there is no difference between the stochastic interpretations.

2. Data

The paleoclimate data studied is a time series of the δ^{18}O in permille as measured in the Greenlandic ice-core extracted as part of the North GReenland Ice-core Project (NGRIP) [45]. The 20-year average values

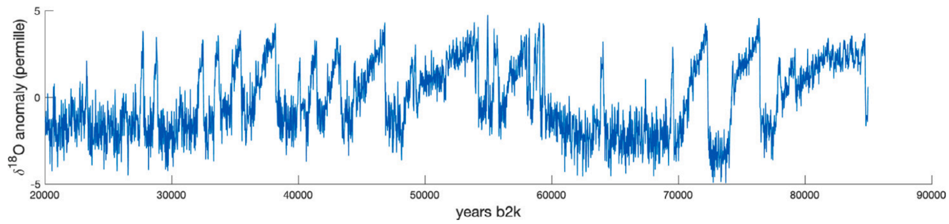

Fig. 1. Detrended δ^{18}O signal from NGRIP from years 20 to 85 kyr b2k (note that time runs from right to left).

on the GICC05modeltext time scale are used [46]. The time series is also truncated at 85 kyr b2k as the resolution decreases further back in time due to the thinning of layers of ice in the ice core. For a further distinction of the two states, the data is detrended. Insolation trends due to orbital variations are removed through subtracting a 25 kyr running mean [17] and the resulting δ^{18}O anomaly is analyzed. This 25 kyr running mean corresponds to the highest frequency of orbital variations, namely precession, which has a period of approximately 20 kyr. This method is effectively a rectangular kernel, which has the important property of not filtering out impulses, i.e. the D–O events themselves. Fig. 1 shows the time series that is the starting point of this study.

3. Methods

3.1. Derivation of multiplicative noise $\sigma(x)$

A heuristic method is used to derive the multiplicative noise term $\sigma(x)$ from the data. Since the fluctuations are larger in the stadials than in the interstadials [40] we prescribe a linearly decreasing function of $\sigma(x)$ with respect to the $\delta^{18}O$ anomaly. Physically, if the noise term is to represent the influence of the atmosphere, an increase in Greenlandic temperatures corresponds to a decrease in the meridional temperature gradient which in turn decreases atmospheric forcing. Following the definition of the stadial and interstadial periods [47], the data is separated into the two states. The values of $\langle x \rangle$ and σ in each of these two states is derived, and the linear function is constructed from these values. To measure σ it is assumed that the signal in either of the two states follows an Ornstein–Uhlenbeck (O–U) process, following [40]

$$dx = -\theta x + \sigma dW_t. \tag{10}$$

For a stationary O–U process the variance is given by the fluctuation–dissipation relation

$$\mathrm{Var}(x) = \frac{\sigma^2}{2\theta}, \tag{11}$$

and the term θ may be recovered from the autocorrelation

$$\langle x(t)x(s)\rangle = \mathrm{Var}(x)\exp[-\theta|t-s|]. \tag{12}$$

3.2. Derivation of the non-linear potentials F(x)

Once the function $\sigma(x)$ is determined the multiplicative Fokker–Planck Eq. (7) is solved for $F(x)$ for the two stochastic calculi. Fig. 2 shows the two potentials $U(x) = \int F(x)dx$ in the Itô and Stratonovich interpretations, along with the potential for the additive noise case (2). When comparing the drift in the multiplicative noise cases to that of the additive noise case, the stability of the interstadial is much reduced. Further, for the Itô case, the interstadial has in fact lost stability such that the resulting climate potential is mono stable.

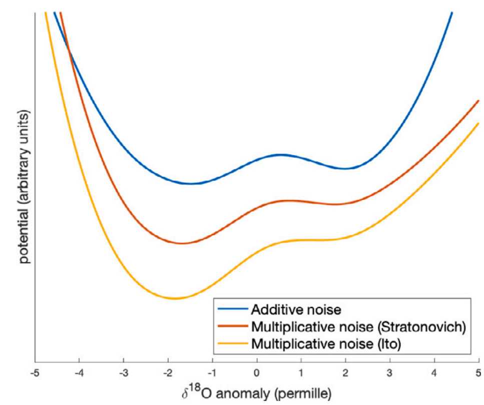

Fig. 2. Derived potentials.

4. Discussion

Due to the qualitative difference of the potential obtained by the two different stochastic interpretations, careful consideration is required when deriving physical properties of the system from observations. The question to be answered is whether the climate potential that underlies the D–O events is monostable, as in the Itô calculus, or bistable, as in the Stratonovich calculus. How is a choice of a stochastic integral made? The data itself is agnostic to interpretation, and mathematically the problem is inconsequential: either interpretation works, as long as they are applied consistently. The choice is ultimately a modeling problem. In this case the SDE inversely modeled by the data requires a conceptual physical model of the phenomenon by which to interpret the results. Conventional understanding is that a physical system is better suited to Stratonovich interpretation. The Wong–Zakai theorem [48] gives that the limit of a sequence of stochastic processes with finite autocorrelation that goes to zero is interpreted as Stratonovich. In this sense it is derived from a continuous process. For a fully discrete system, for example in financial analysis, the Itô interpretation is more appropriate.

However, the one dimensional SDE is generally the result of simplification of dynamics that occur on multiple time scales. Through this reduction, multiplicative noise of the Itô type can be seen in physical systems as well. The most prominent example are inertial systems with colored noise,

$$\begin{aligned} \ddot{x} &= -\gamma(x)\dot{x} + F(x) + \sigma(x)\eta \\ \dot{\eta} &= -\frac{a}{\tau_n}\eta + \frac{\lambda}{\tau_n}\xi_t \end{aligned} \tag{13}$$

where for convenience, we use the notation, $\dot{x} \equiv dx/dt$ and $dW_t = \xi_t dt$.

To reduce the complexity of Eq. (13), two limits are taken: one is an adiabatic elimination of fast inertia ($\tau_r = \gamma(x)^{-1} \to 0$) into the overdamped regime (also known as the strong dissipation or Smoluchowski regime) and the second is the white-noise approximation of the unresolved dynamics ($\tau_n \to 0$). If the time scale of the inertial relaxation is greater than that of the noise autocorrelation ($\tau_r \gg \tau_n$), then the multiplicative noise is Itô in the limit as both go to zero. If the noise autocorrelation in greater than the relaxation time scale of the inertia ($\tau_r \ll \tau_n$), the Stratonovich interpretation for the multiplicative noise is used [49–54].

The system of Eq. (13) in the white noise limit is

$$\ddot{x} = -\gamma(x)\dot{x} + F(x) + \sigma(x)\xi_t. \tag{14}$$

For additive noise $\sigma(x) = \sigma$, adiabatic elimination is equivalent to setting the left hand side of Eq. (14) equal to zero [55,56]. This is not the case for multiplicative noise, where the limits must be taken carefully. Eq. (13) in the white noise and adiabatic limit is

$$\dot{x} = \frac{F(x)}{\gamma(x)} + \frac{1}{2}\frac{\sigma(x)^2}{\gamma(x)}\frac{\partial\gamma(x)^{-1}}{\partial x} - \frac{\alpha}{2}\frac{\partial\sigma(x)^2\gamma(x)^{-2}}{\partial x} + \frac{\sigma(x^*)}{\gamma(x^*)}\xi_t, \tag{15}$$

where $x^* = \alpha x(t+dt) + (1-\alpha)x(t)$ [56]. In the case where the fluctuation–dissipation theorem applies,

$$\frac{\sigma(x)^2}{2\gamma(x)} = \text{constant} \tag{16}$$

the system obtained by setting the left hand side of Eq. (13) to zero

$$\dot{x} = \frac{F(x)}{\gamma(x)} + \frac{\sigma(x^*)}{\gamma(x^*)}\xi_t, \tag{17}$$

is equivalent to the system interpreted in the *anti-Itô* ($\alpha = 1$) sense. This is the result of Volpe et al. [57] and Lançon et al. [58], see also [56,59].

Another example where we see Itô multiplicative noise is in an SDE with noise feedback delay [60]. This is a circuit system which has been designed to effectively act in an Itô manner by implementing an explicit dependence of the multiplicative noise on the previous time step through introducing a delay of the feedback of the state on the noise term, i.e. the x in the $\sigma(x)$ term. One dimensional systems can also be either Itô or Stratonovich in the case where multiple times scales are involved [61,62].

Now understanding the situation, we return to the case of the δ^{18}O anomaly time series. To accompany the SDE derived directly from the data, a conceptual physical model is required, which will not only provide an idea of the underlying mechanisms but also provide a stochastic interpretation. The stochastic bistable Stommel-type model [36] has been extended to include multiplicative noise by Timmermann, Lohmann and Monahan [63,64]. Their multiplicative noise term arises due to a stochastically parameterized eddy transport in salinity and temperature. As in Cessi [36], the temperature relaxes quickly to some mean value, and a white noise approximation may be made. In this case the white-noise limit is taken via the Wong–Zakai theorem, so that the resulting SDE uses Stratonovich calculus. Then the thermohaline circulation remains bistable based on the model derived from the data.

In another study, Kwasniok and Lohmann [65] fit the D–O event time series data to a stochastic oscillator

$$\ddot{x} = -\gamma\dot{x} + F(x) + \sigma\xi_t \tag{18}$$

which is Eq. (14) with constant damping and additive noise. The variable x represents a temperature proxy and its derivative is the change in temperature, but otherwise the system is not physically defined. They find it is in the strongly dissipative regime and can be reduced to a first-order equation by adiabatic elimination of the second derivative. By augmenting the system with multiplicative noise, a reduction to first order would result in an SDE

$$dx = \frac{F(x)}{\gamma}dt + \frac{\sigma(x)}{\gamma}dW_t, \tag{19}$$

with the Itô interpretation. However, this interpretation still relies on the fact that the autocorrelation of the noise is assumed to be decaying faster than the relaxation of the fast variable, i.e $\tau_r \gg \tau_n$. This could be understood as a temporal scale of the thermal relaxation time of the surface temperature of Greenland and the autocorrelation time of the atmospheric variability. The stochastic interpretation is still a result of modeling choices.

Various studies model the D–O events using a two-dimensional model generally assume the form of a van der Pol or FitzHugh–Nagumo type model [15,17,22]. These have a form similar to Eq. (14), but the variable of interest is on the fast timescale. This means the system is in the underdamped regime and cannot be adiabatically reduced to one dimension. While the models in these studies include only additive noise, we again consider their stochastic interpretation in the possible case of multiplicative noise. In the underdamped regime of Eq. (14) the difference between the Itô and Stratonovich integrals is smaller than dt, so they are equivalent. First, the second order SDE is split into a system of first-order equations

$$\begin{aligned} dx &= v\,dt \\ dv &= \big(-\gamma v + F(x)\big)dt + \sigma(x)dW_t. \end{aligned} \tag{20}$$

Using the α-convention, $x^* = (1-\alpha)x(t) + \alpha x(t+dt)$. Expanding the term $\sigma(x^*)dW_t$,

$$\begin{aligned} \sigma(x^*)dW_t &\approx \sigma(x(t))dW_t + \sigma'(x(t))\big(\alpha x(t+dt) + (1-\alpha)x(t) - x(t)\big) \\ &\quad \times dW_t + \mathcal{O}(dt^2) \\ &\approx \sigma(x(t))dW_t + \alpha\sigma'(x(t))dx\,dW_t + \mathcal{O}(dt^2) \\ &\approx \sigma(x(t))dW_t + \alpha\sigma'(x(t))v(t)dt\,dW_t + \mathcal{O}(dt^2), \end{aligned} \tag{21}$$

and the difference between the interpretations given in the α term is of the order $(dt)^{3/2} < dt$ and thus vanishes faster than the time scale of the dynamics. If instead there is a stochastic component to both variables,

$$\begin{aligned} dx &= v\,dt + \sigma_x dW_t \\ dv &= \big(-\gamma v + F(x)\big)dt + \sigma_v(x)dW_t. \end{aligned} \tag{22}$$

Then the stochastic term is in the fast variable is instead

$$\sigma_v(x^*)dW_t \approx \sigma_v\big(x(t)\big)dW_t + \alpha\sigma_v'\big(x(t)\big)v(t)dt\,dW_t + \alpha\sigma_v'(x(t))\sigma_x dt + \mathcal{O}(dt^2), \tag{23}$$

and the difference between the Itô and Stratonovich interpretations is now of the order dt and must be taken into account.

The previously mentioned study of Krumscheid et al. [44] has state-dependent noise in a data-driven model derivation and assumes Itô calculus. Their multiplicative noise term takes the form of a piecewise constant function, and as such has zero derivative except at the jump. Due to this, the difference between the Itô and Stratonovich interpretations vanishes except at a single point, which can be safely disregarded. Their study does include continuous functions as candidate noise terms but find the additive noise model outperforms them with regard to their parameter fitting routine.

5. Conclusions

In this study, we have derived a data-driven conceptual model of the D–O events including multiplicative noise and seen the reduced stability of the interstadial state when compared to the stadial. We also describe the need to specify a stochastic calculus to be able to interpret the climate potential of the model. We have outlined models which may be interpreted as either Itô or Stratonovich in limiting cases and suggest that the interpretation depends on a physical understanding of the system. For example, if the system derived from the data were meant to represent the stability of the thermohaline circulation but the stochastic integral interpreted as Itô, one would arrive to the erroneous conclusion that the data shows the overturning circulation was monostable, when there is evidence beyond the ice-core record that it is bistable.

The result that the Itô interpretation leads to monostable dynamics is one that is echoed by other conceptual models [17,33,65]. These monostable excitable models with fast–slow dynamics, which require at least 2 dimensions, mirrors the scenario in which the Itô interpretation is applicable. That is, the Itô interpretation comes about due to reduction of a strongly dissipative inertial system, which can be represented as a 2D system with different timescales, to a system in 1D.

Ultimately, this work shows that when deriving stochastic models, the stochastic interpretation can fundamentally affect the results. Interpreting whether the underlying nature of the climate that gave rise to D–O events is a mono- or bi-stable system is vital step to understanding the phenomenon. This is affected by the noise interpretation, which seems to be a completely non-physical, mathematical formalism but is in fact determined by the timescales of the dynamics of the system.

CRediT authorship contribution statement

Kolja Kypke: Investigation, Writing – original draft, Conceptualization. **Peter Ditlevsen:** Conceptualization, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The ice-core record from the NGRIP project is available at https://www.iceandclimate.nbi.ku.dk/data/.

Acknowledgments

K.K. is supported by the European Union's Horizon 2020 project under the Marie Skłodowska-Curie Innovative Training Network CriticalEarth, grant agreement no. 956170. This is a contribution to the European Union's Horizon 2020 research and innovation programme Tipping Points in the Earth System (TiPES), grant agreement no. 820970.

References

[1] W. Dansgaard, S.J. Johnsen, H.B. Clausen, D. Dahl-Jensen, N.S. Gundestrup, C.U. Hammer, C.S. Hvidberg, J.P. Steffensen, A.E. Sveinbjörnsdottir, J. Jouzel, G. Bond, Evidence for general instability of past climate from a 250-kyr ice-core record, Nature 364 (6434) (1993) 218–220, http://dx.doi.org/10.1038/364218a0.

[2] P. Kindler, M. Guillevic, M. Baumgartner, J. Schwander, A. Landais, M. Leuenberger, Temperature reconstruction from 10 to 120 kyr b2k from the NGRIP ice core, Clim. Past 10 (2) (2014) 887–902, http://dx.doi.org/10.5194/cp-10-887-2014.

[3] S. Barker, G. Knorr, R.L. Edwards, F. Parrenin, A.E. Putnam, L.C. Skinner, E. Wolff, M. Ziegler, 800,000 Years of abrupt climate variability, Science 334 (2011) 347, http://dx.doi.org/10.1126/science.1203580.

[4] B. Martrat, J.O. Grimalt, N.J. Shackleton, L. De Abreu, M.A. Hutterli, T.F. Stocker, Four climate cycles of recurring deep and surface water destabilizations on the Iberian margin, Science 317 (5837) (2007) 502–507, http://dx.doi.org/10.1126/science.1139994.

[5] F. Held, H. Cheng, R.L. Edwards, O. Tüysüz, K. Koç, D. Fleitmann, Dansgaard-Oeschger cycles of the penultimate and last glacial period recorded in stalagmites from Türkiye, Nature Commun. 15 (1) (2024) 1183, http://dx.doi.org/10.1038/s41467-024-45507-5.

[6] T. Mitsui, M. Crucifix, Influence of external forcings on abrupt millennial-scale climate changes: a statistical modelling study, Clim. Dyn. 48 (7) (2017) 2729–2749, http://dx.doi.org/10.1007/s00382-016-3235-z.

[7] J.A. Rial, R. Saha, Modeling abrupt climate change as the interaction between sea ice extent and mean ocean temperature under orbital insolation forcing, in: Abrupt Climate Change: Mechanisms, Patterns, and Impacts, American Geophysical Union (AGU), 2011, pp. 57–74, http://dx.doi.org/10.1029/2010GM001027.

[8] K. Riechers, T. Mitsui, N. Boers, M. Ghil, Orbital insolation variations, intrinsic climate variability, and Quaternary glaciations, Clim. Past 18 (4) (2022) 863–893, http://dx.doi.org/10.5194/cp-18-863-2022.

[9] M. Crucifix, Oscillators and relaxation phenomena in Pleistocene climate theory, Phil. Trans. R. Soc. A 370 (1962) (2012) 1140–1165, http://dx.doi.org/10.1098/rsta.2011.0315.

[10] M. Schulz, A. Paul, A. Timmermann, Relaxation oscillators in concert: A framework for climate change at millennial timescales during the late Pleistocene, Geophys. Res. Lett. 29 (24) (2002) 46–1–46–4, http://dx.doi.org/10.1029/2002GL016144.

[11] A. Ganopolski, S. Rahmstorf, Rapid changes of glacial climate simulated in a coupled climate model, Nature 409 (6817) (2001) 153–158, http://dx.doi.org/10.1038/35051500.

[12] P.D. Ditlevsen, Observation of α-stable noise induced millennial climate changes from an ice-core record, Geophys. Res. Lett. 26 (10) (1999) 1441–1444, http://dx.doi.org/10.1029/1999GL900252.

[13] P.D. Ditlevsen, M.S. Kristensen, K.K. Andersen, The recurrence time of Dansgaard–Oeschger events and limits on the possible periodic component, J. Clim. 18 (14) (2005) 2594–2603, http://dx.doi.org/10.1175/JCLI3437.1.

[14] P.D. Ditlevsen, K.K. Andersen, A. Svensson, The DO-climate events are probably noise induced: statistical investigation of the claimed 1470 years cycle, Clim. Past 3 (1) (2007) 129–134, http://dx.doi.org/10.5194/cp-3-129-2007.

[15] F. Kwasniok, Analysis and modelling of glacial climate transitions using simple dynamical systems, Philos. Trans. R. Soc. Lond. Ser. A: Math. Phys. Eng. Sci. 371 (1991) (2013) 20110472, http://dx.doi.org/10.1098/rsta.2011.0472.

[16] V.N. Livina, F. Kwasniok, T.M. Lenton, Potential analysis reveals changing number of climate states during the last 60 kyr, Clim. Past 6 (1) (2010) 77–82, http://dx.doi.org/10.5194/cp-6-77-2010.

[17] J. Lohmann, P.D. Ditlevsen, A consistent statistical model selection for abrupt glacial climate changes, Clim. Dyn. 52 (11) (2019) 6411–6426, http://dx.doi.org/10.1007/s00382-018-4519-2.

[18] A. Ganopolski, S. Rahmstorf, Abrupt glacial climate changes due to stochastic resonance, Phys. Rev. Lett. 88 (3) (2002) 038501, http://dx.doi.org/10.1103/PhysRevLett.88.038501.

[19] T.M. Dokken, K.H. Nisancioglu, C. Li, D.S. Battisti, C. Kissel, Dansgaard-Oeschger cycles: Interactions between ocean and sea ice intrinsic to the Nordic seas, Paleoceanography 28 (3) (2013) 491–502, http://dx.doi.org/10.1002/palo.20042.

[20] S.V. Petersen, D.P. Schrag, P.U. Clark, A new mechanism for Dansgaard-Oeschger cycles, Paleoceanography 28 (1) (2013) 24–30, http://dx.doi.org/10.1029/2012PA002364.

[21] R. Banderas, J. Álvarez-Solas, M. Montoya, Role of CO_2 and Southern Ocean winds in glacial abrupt climate change, Clim. Past 8 (3) (2012) 1011–1021, http://dx.doi.org/10.5194/cp-8-1011-2012.

[22] G. Vettoretti, P. Ditlevsen, M. Jochum, S.O. Rasmussen, Atmospheric CO2 control of spontaneous millennial-scale ice age climate oscillations, Nat. Geosci. 15 (4) (2022) 300–306, http://dx.doi.org/10.1038/s41561-022-00920-7.

[23] J. Lohmann, A. Svensson, Ice core evidence for major volcanic eruptions at the onset of Dansgaard–Oeschger warming events, Clim. Past 18 (9) (2022) 2021–2043, http://dx.doi.org/10.5194/cp-18-2021-2022.

[24] K. Sakai, W.R. Peltier, Dansgaard–Oeschger oscillations in a coupled atmosphere–ocean climate model, J. Clim. 10 (5) (1997) 949–970, http://dx.doi.org/10.1175/1520-0442(1997)010<0949:DOOIAC>2.0.CO;2.

[25] G. Vettoretti, W.R. Peltier, Fast physics and slow physics in the nonlinear Dansgaard–Oeschger relaxation oscillation, J. Clim. 31 (9) (2018) 3423–3449, http://dx.doi.org/10.1175/JCLI-D-17-0559.1.

[26] P.U. Clark, N.G. Pisias, T.F. Stocker, A.J. Weaver, The role of the thermohaline circulation in abrupt climate change, Nature 415 (6874) (2002) 863–869, http://dx.doi.org/10.1038/415863a.

[27] A.C. Clement, L.C. Peterson, Mechanisms of abrupt climate change of the last glacial period, Rev. Geophys. 46 (4) (2008) http://dx.doi.org/10.1029/2006RG000204.

[28] C. Li, A. Born, Coupled atmosphere-ice-ocean dynamics in Dansgaard-Oeschger events, Quat. Sci. Rev. 203 (2019) 1–20, http://dx.doi.org/10.1016/j.quascirev.2018.10.031.

[29] F. Kwasniok, G. Lohmann, Deriving dynamical models from paleoclimatic records: Application to glacial millennial-scale climate variability, Phys. Rev. E 80 (6) (2009) 066104, http://dx.doi.org/10.1103/PhysRevE.80.066104.

[30] N. Boers, M.D. Chekroun, H. Liu, D. Kondrashov, D.-D. Rousseau, A. Svensson, M. Bigler, M. Ghil, Inverse stochastic–dynamic models for high-resolution Greenland ice core records, Earth Syst. Dyn. 8 (4) (2017) 1171–1190, http://dx.doi.org/10.5194/esd-8-1171-2017.

[31] K. Riechers, L. Rydin Gorjão, F. Hassanibesheli, P.G. Lind, D. Witthaut, N. Boers, Stable stadial and interstadial states of the last glacial's climate identified in a combined stable water isotope and dust record from Greenland, Earth Syst. Dyn. 14 (3) (2023) 593–607, http://dx.doi.org/10.5194/esd-14-593-2023.

[32] N. Boers, M. Ghil, D.-D. Rousseau, Ocean circulation, ice shelf, and sea ice interactions explain Dansgaard-Oeschger cycles, Proc. Natl. Acad. Sci. USA 115 (47) (2018) E11005–E11014, http://dx.doi.org/10.1073/pnas.1802573115.

[33] K. Riechers, G. Gottwald, N. Boers, Glacial abrupt climate change as a multi-scale phenomenon resulting from monostable excitable dynamics, J. Clim. (2024) http://dx.doi.org/10.1175/JCLI-D-23-0308.1.

[34] G. Bond, W. Broecker, S. Johnsen, J. McManus, L. Labeyrie, J. Jouzel, G. Bonani, Correlations between climate records from North Atlantic sediments and Greenland ice, Nature 365 (6442) (1993) 143–147, http://dx.doi.org/10.1038/365143a0.

[35] W.S. Broecker, D.M. Peteet, D. Rind, Does the ocean-atmosphere system have more than one stable mode of operation? Nature 315 (6014) (1985) 21–26, http://dx.doi.org/10.1038/315021a0.

[36] P. Cessi, A simple box model of stochastically forced thermohaline flow, J. Phys. Oceanogr. 24 (9) (1994) 1911–1920, http://dx.doi.org/10.1175/1520-0485(1994)024<1911:ASBMOS>2.0.CO;2.

[37] H. Stommel, Thermohaline convection with two stable regimes of flow, Tellus 13 (2) (1961) 224–230, http://dx.doi.org/10.3402/tellusa.v13i2.9491.

[38] S. Rahmstorf, Bifurcations of the Atlantic thermohaline circulation in response to changes in the hydrological cycle, Nature 378 (6553) (1995) 145–149, http://dx.doi.org/10.1038/378145a0.

[39] I. Malmierca-Vallet, L.C. Sime, the D–O community members, Dansgaard–Oeschger events in climate models: review and baseline Marine Isotope Stage 3 (MIS3) protocol, Clim. Past 19 (5) (2023) 915–942, http://dx.doi.org/10.5194/cp-19-915-2023.

[40] P.D. Ditlevsen, S. Ditlevsen, K.K. Andersen, The fast climate fluctuations during the stadial and interstadial climate states, Ann. Glaciol. 35 (1) (2002) 457–462, http://dx.doi.org/10.3189/172756402781816870.

[41] K. Itô, Stochastic integral, Proc. Imp. Acad. 20 (8) (1944) 519–524, http://dx.doi.org/10.3792/pia/1195572786.

[42] R.L. Stratonovich, A new representation for stochastic integrals and equations, SIAM J. Control 4 (2) (1966) 362–371, http://dx.doi.org/10.1137/0304028.

[43] N.G. van Kampen, Itô versus Stratonovich, J. Stat. Phys. 24 (1) (1981) 175–187, http://dx.doi.org/10.1007/BF01007642.

[44] S. Krumscheid, M. Pradas, G.A. Pavliotis, S. Kalliadasis, Data-driven coarse graining in action: Modeling and prediction of complex systems, Phys. Rev. E 92 (4) (2015) 042139, http://dx.doi.org/10.1103/PhysRevE.92.042139.

[45] North Greenland Ice Core Project members, High-resolution record of Northern Hemisphere climate extending into the last interglacial period, Nature 431 (7005) (2004) 147–151, http://dx.doi.org/10.1038/nature02805.

[46] S.O. Rasmussen, M. Bigler, S.P. Blockley, T. Blunier, S.L. Buchardt, H.B. Clausen, I. Cvijanovic, D. Dahl-Jensen, S.J. Johnsen, H. Fischer, V. Gkinis, M. Guillevic, W.Z. Hoek, J.J. Lowe, J.B. Pedro, T. Popp, I.K. Seierstad, J.P. Steffensen, A.M. Svensson, P. Vallelonga, B.M. Vinther, M.J. Walker, J.J. Wheatley, M. Winstrup, A stratigraphic framework for abrupt climatic changes during the Last Glacial period based on three synchronized Greenland ice-core records: refining and extending the INTIMATE event stratigraphy, Quat. Sci. Rev. 106 (2014) 14–28, http://dx.doi.org/10.1016/j.quascirev.2014.09.007.

[47] J. Lohmann, P.D. Ditlevsen, Objective extraction and analysis of statistical features of Dansgaard–Oeschger events, Clim. Past 15 (5) (2019) 1771–1792, http://dx.doi.org/10.5194/cp-15-1771-2019.

[48] E. Wong, M. Zakai, On the convergence of ordinary integrals to stochastic integrals, Ann. Math. Stat. 36 (5) (1965) 1560–1564, http://dx.doi.org/10.1214/aoms/1177699916.

[49] S. Faetti, C. Festa, L. Fronzoni, P. Grigolini, P. Martano, Multiplicative stochastic processes in nonlinear systems: Noise-induced transition from the overdamped to the inertial regime, Phys. Rev. A 30 (6) (1984) 3252–3263, http://dx.doi.org/10.1103/PhysRevA.30.3252.

[50] R. Graham, A. Schenzle, Stabilization by multiplicative noise, Phys. Rev. A 26 (3) (1982) 1676–1685, http://dx.doi.org/10.1103/PhysRevA.26.1676.

[51] S. Hottovy, A. McDaniel, G. Volpe, J. Wehr, The Smoluchowski-Kramers limit of stochastic differential equations with arbitrary state-dependent friction, Comm. Math. Phys. 336 (3) (2015) 1259–1283, http://dx.doi.org/10.1007/s00220-014-2233-4.

[52] R. Kupferman, G.A. Pavliotis, A.M. Stuart, Itô versus Stratonovich white-noise limits for systems with inertia and colored multiplicative noise, Phys. Rev. E 70 (3) (2004) 036120, http://dx.doi.org/10.1103/PhysRevE.70.036120.

[53] G.A. Pavliotis, A.M. Stuart, Analysis of white noise limits for stochastic systems with two fast relaxation times, Multiscale Model. Simul. 4 (1) (2005) 1–35, http://dx.doi.org/10.1137/040610507.

[54] K. Sekimoto, Temporal coarse graining for systems of Brownian particles with non-constant temperature, J. Phys. Soc. Japan 68 (4) (1999) 1448–1449, http://dx.doi.org/10.1143/JPSJ.68.1448.

[55] C.W. Gardiner, Handbook of Stochastic Methods: For Physics, Chemistry and Natural Sciences, second ed., in: Springer Series in Synergetics, vol. 13, Springer, Berlin, 1997.

[56] T. Kuroiwa, K. Miyazaki, Brownian motion with multiplicative noises revisited, J. Phys. A 47 (1) (2014) 012001, http://dx.doi.org/10.1088/1751-8113/47/1/012001.

[57] G. Volpe, L. Helden, T. Brettschneider, J. Wehr, C. Bechinger, Influence of noise on force measurements, Phys. Rev. Lett. 104 (17) (2010) 170602, http://dx.doi.org/10.1103/PhysRevLett.104.170602.

[58] P. Lançon, G. Batrouni, L. Lobry, N. Ostrowsky, Drift without flux: Brownian walker with a space-dependent diffusion coefficient, Europhys. Lett. (EPL) 54 (1) (2001) 28–34, http://dx.doi.org/10.1209/epl/i2001-00103-6.

[59] R. Mannella, P.V.E. McClintock, Itô versus Stratonovich: 30 years later, Fluct. Noise Lett. 11 (1) (2012) 1240010, http://dx.doi.org/10.1142/S021947751240010X.

[60] G. Pesce, A. McDaniel, S. Hottovy, J. Wehr, G. Volpe, Stratonovich-to-Itô transition in noisy systems with multiplicative feedback, Nature Commun. 4 (1) (2013) 2733, http://dx.doi.org/10.1038/ncomms3733.

[61] W. Moon, J.S. Wettlaufer, On the interpretation of Stratonovich calculus, New J. Phys. 16 (5) (2014) 055017, http://dx.doi.org/10.1088/1367-2630/16/5/055017.

[62] T. Morita, On the interpretation of multiplicative white noise, Phys. Lett. A 82 (5) (1981) 215–217, http://dx.doi.org/10.1016/0375-9601(81)90187-0.

[63] A. Timmermann, G. Lohmann, Noise-induced transitions in a simplified model of the thermohaline circulation, J. Phys. Oceanogr. 30 (8) (2000) 1891–1900, http://dx.doi.org/10.1175/1520-0485(2000)030<1891:NITIAS>2.0.CO;2.

[64] A.H. Monahan, A. Timmermann, G. Lohmann, Comments on "Noise-induced transitions in a simplified model of the thermohaline circulation", J. Phys. Oceanogr. 32 (3) (2002) 1112–1116, http://dx.doi.org/10.1175/1520-0485(2002)032<1112:CONITI>2.0.CO;2.

[65] F. Kwasniok, G. Lohmann, A stochastic nonlinear oscillator model for glacial millennial-scale climate transitions derived from ice-core data, Nonlinear Process. Geophys. 19 (6) (2012) 595–603, http://dx.doi.org/10.5194/npg-19-595-2012.